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# ON THE THEORY OF INHOMOGENEOUS ELECTROELASTIC PLATES

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A method of constructing a certain class of exact inhomogeneous solutions for transversely inhomogeneous electroelastic plates is presented. Two cases are considered: 1) the plate is a piezoactive dielectric material whose electric and elastic properties vary across its thickness according to some arbitrary law, and 2) the plate consists of alternate metal and piezoactive dielectric layers.

The construction of homogeneous solutions for the first case was considered in [1], and for homogeneous plates such solutions were derived in [2]. In [3] Gol'denveizer's asymptotic method was used for obtaining inhomogeneous solutions for piezoelectric plates homogeneous across their thickness.

**1.** Let us consider the electroelastic equilibrium of a plate occupying region $\Omega = S \times [-h, h]$, where $S$ is the median surface, $2h$ is the thickness, $\Gamma = \partial S \times [-h, h]$ is the side face, $\partial S$ is the boundary of $S$, and $a$ is a characteristic linear dimension of $S$. We relate the plate to a Cartesian system of coordinates $(x_1, x_2, x_3)$ with origin in $S$ and the $x_3$-axis normal to $S$. We assume that the properties of the plate material are defined by the following relations [4, 5]:

$$\begin{aligned}
\sigma_{11} &= c_{11}S_{11} + c_{12}S_{22} + c_{13}S_{33} - e_{31}E_3 \\
\sigma_{22} &= c_{12}S_{11} + c_{11}S_{22} + c_{13}S_{33} - e_{31}E_3 \\
\sigma_{33} &= c_{13}S_{11} + c_{13}S_{22} + c_{33}S_{33} - e_{33}E_3 \\
\sigma_{12} &= 2c_{66}S_{12} = (c_{11} - c_{12})S_{12} \\
\sigma_{\alpha 3} &= 2c_{44}S_{\alpha 3} - e_{15}E_\alpha, \quad D_\alpha = 2e_{15}S_{\alpha 3} + \varepsilon_{11}E_\alpha \quad (\alpha = 1, 2) \\
D_3 &= e_{31}S_{11} + e_{31}S_{22} + e_{33}S_{33} + \varepsilon_{33}E_3
\end{aligned} \tag{1.1}$$

where the moduli of elasticity $c_{ij}$, the piezoelectric moduli $e_{kl}$, and the permittivities $\varepsilon_{mm}$ are arbitrary piecewise continuous functions $\zeta\ (x_3 = h\zeta)$. Note that functions $e_{kl}\ (\zeta)$ may vanish on individual connected sections of variation of coordinate $\zeta$, which indicates absence of piezoelectric effects in respective layers.

The equations of electroelastic equilibrium for a medium of the described type can be written in vector form

$$\partial\,(M_0\partial \mathbf{U}) + \varepsilon\,[\partial\,(M_1\mathbf{U}) + M_1{}^*\partial\mathbf{U}] + \varepsilon^2 M_2\mathbf{U} = 0 \tag{1.2}$$

$$\mathbf{U} = \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \\ \varphi \end{Vmatrix}, \quad M_0 = \begin{Vmatrix} c_{44} & 0 & 0 & 0 \\ 0 & c_{44} & 0 & 0 \\ 0 & 0 & c_{33} & e_{33} \\ 0 & 0 & e_{33} & -\varepsilon_{33} \end{Vmatrix}$$

$$M_1 = \begin{Vmatrix} 0 & 0 & c_{44}\partial_1 & d_{15}\partial_1 \\ 0 & 0 & c_{44}\partial_2 & e_{15}\partial_2 \\ c_{13}\partial_1 & c_{13}\partial_2 & 0 & 0 \\ e_{31}\partial_1 & e_{31}\partial_2 & 0 & 0 \end{Vmatrix},$$

$$M_2 = \begin{Vmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 & (c_{12}+c_{66})\partial_1\partial_2 & 0 & 0 \\ (c_{12}+c_{66})\partial_1\partial_2 & c_{66}\partial_1^2 + c_{11}\partial_2^2 & 0 & 0 \\ 0 & 0 & c_{44}\Delta & e_{15}\Delta \\ 0 & 0 & e_{15}\Delta & -\varepsilon_{11}\Delta \end{Vmatrix}$$

$$\partial_\alpha = \frac{\partial}{\partial\xi_\alpha}, \quad \partial = \frac{\partial}{\partial\zeta}, \quad \xi_\alpha = \frac{x_\alpha}{a}, \quad \zeta = \frac{x_3}{h}$$

$$\varepsilon = \frac{h}{a}, \quad \Delta = \partial_1^2 + \partial_2^2 \quad (\alpha = 1, 2)$$

where $u_i$ are displacements, $\varphi$ is the electric field potential, and $M_1^*$ is the transpose of matrix $M_1$.

We assume that at the plate end faces the following boundary conditions apply:

$$\sigma_{k3}(\xi, \pm1) = q_k^\pm(\xi), \quad D_3(\xi, \pm1) = d^\pm(\xi), \quad \xi = (\xi_1, \xi_2), \quad k = 1, 2, 3 \tag{1.3}$$

Boundary conditions at the $\Gamma$ plate side face are not defined in this case, since below we deal with the derivation of particular solutions of Eqs. (1.2) that satisfy boundary conditions (1.3). These solutions, which we shall call inhomogeneous, are derived for the case when functions $q_k^\pm(\xi)$ and $d^\pm(\xi)$ are polyharmonic. Taking into account that virtually any smooth function can be approximated by a polyharmonic one (e.g., by a polynomial), such inhomogeneous solution used in conjuction with the homogeneous ones derived in [1] enable us to solve effectively a fairly wide range of boundary value problems for plates of finite dimensions.

It, is moreover, possible to use exact solutions of the three-dimensional problem for analyzing the errors of solutions based on the applied theories considered in [6—8] and others.

**2.** We introduce the subsidiary relations

$$u_1 = \varepsilon(\partial_1 a_1 + \partial_2 a_2), \quad u_2 = \varepsilon(\partial_2 a_1 - \partial_1 a_2) \tag{2.1}$$

$$q_1^\pm = \varepsilon(\partial_1\tau_1^\pm + \partial_2\tau_2^\pm), \quad q_2^\pm = \varepsilon(\partial_2\tau_1^\pm - \partial_1\tau_2^\pm) \tag{2.2}$$

The substitution of (2.1) and (2.2) into (1.2) and (1.3) makes it possible to divide the input problem into two, as follows:

$$L(\varepsilon^2\Delta)\mathbf{V} \equiv L_0\mathbf{V} + \varepsilon^2\Delta L_1\mathbf{V} = 0 \tag{2.3}$$
$$N(\varepsilon^2\Delta)\mathbf{V}\,|_{\zeta=\pm1} \equiv (N_0\mathbf{V} + \varepsilon^2\Delta N_1\mathbf{V})_{\zeta=\pm1} = \boldsymbol{\sigma}^\pm$$

$$H(\varepsilon^2\Delta)a_2 = \partial c_{44}\partial a_2 + c_{66}\varepsilon^2\Delta a_2 = 0 \tag{2.4}$$
$$(c_{44}\partial a_2)_{\zeta=\pm1} = \tau_2^\pm$$

where $\mathbf{V} = (a_1, u_3, \varphi)$ and $\boldsymbol{\sigma}^\pm = (\tau_1^\pm, q_3^\pm, d^\pm)$ are vector functions, and $L_\alpha$ and $N_\alpha$ are metric operators of the form

$$L_0 = \left\| \begin{matrix} \partial c_{44}\partial & c_{13}\partial + \partial c_{44} & e_{31}\partial + \partial e_{15} \\ 0 & \partial c_{33}\partial & \partial e_{33}\partial \\ 0 & \partial e_{33}\partial & -\partial \varepsilon_{33}\partial \end{matrix} \right\|,$$

$$L_1 = \left\| \begin{matrix} c_{11} & 0 & 0 \\ c_{44}\partial + \partial c_{13} & c_{44} & e_{15} \\ e_{15}\partial + \partial e_{31} & e_{15} & -\varepsilon_{11} \end{matrix} \right\|$$

$$N_0 = \left\| \begin{matrix} c_{44}\partial & c_{44} & e_{15} \\ 0 & c_{33}\partial & e_{33}\partial \\ 0 & e_{33}\partial & -\varepsilon_{33}\partial \end{matrix} \right\|, \quad N_1 = \left\| \begin{matrix} 0 & 0 & 0 \\ c_{13} & 0 & 0 \\ e_{31} & 0 & 0 \end{matrix} \right\|$$

First, let us consider problem (2.3) on the assumption that the vector components $\sigma^+$ and $\sigma^-$ are of the form

$$\tau_1^\pm = d_1^\pm m(\xi), \quad q_3^\pm = d_2^\pm m(\xi), \quad d^\pm = d_3^\pm m(\xi) \tag{2.5}$$

where the function $m(\xi)$ satisfies the equation

$$\Delta m + \gamma^2 m = 0 \tag{2.6}$$

We seek a solution of the form

$$\mathbf{V} = \mathbf{X}(\zeta) m(\xi) \tag{2.7}$$

For this we substitute (2.5) and (2.7) into (2.3), and obtain for the determination of vector $\mathbf{X}(\zeta)$ the problem

$$L(-\varepsilon^2\gamma^2)\mathbf{X} = 0, \quad N(-\varepsilon^2\gamma^2)\mathbf{X}\,|_{\zeta=\pm 1} = \mathbf{d}^\pm \tag{2.8}$$
$$\mathbf{d}^\pm = (d_1^\pm, d_2^\pm, d_3^\pm)\|$$

Using the results of [1] it is possible to show that an operator inverse of the operator generated by problem (2.8) when $\gamma = 0$ has a quadruple pole. This enables us to seek the solution of problem (2.8) of the **form**

$$\mathbf{X} = (\varepsilon\gamma)^{-4} \sum_{k=0}^{\infty} (\varepsilon\gamma)^{2k} \mathbf{X}_k \tag{2.9}$$

For Eq. (2.6) we seek a solution of the form

$$m = \gamma^l \sum_{t=0}^{\infty} \gamma^{2t} m_t \tag{2.10}$$

The substitution of (2.10) into (2.6) yields the recurrent formulas

$$\Delta m_0 = 0, \quad \Delta m_1 = -m_0, \ldots, \quad \Delta m_t = -m_{t-1}, \ldots$$

which imply that $\Delta^{t+1} m_t = 0$, hence the coefficients of expansion (2.10) are polyharmonic functions.

Using (2.9) and (2.10) we obtain for vector $\mathbf{V}$ the following expansions:

$$\mathbf{V} = \gamma^{l-4} \sum_{k=0}^{\infty} \gamma^{2k} \mathbf{V}_k \tag{2.11}$$

$$\mathbf{V}_k = \varepsilon^{-4} \sum_{s=0}^{k} \varepsilon^{2s} \mathbf{X}_s m_{k-s} \tag{2.12}$$

We introduce the operator

$$\Pi_{ln}(\cdot) = \frac{1}{(2n+4)!} \lim_{\gamma\to 0} \frac{\partial^{2n+4}}{\partial\gamma^{2n+4}} \gamma^{4-l}(\cdot)$$

which in expansion (2.10) "cuts out" one term of the expansion of $m_n$, i.e. $\Pi_{ln}(m) = m_n$, and is commutative with operators $L(\varepsilon^2\Delta)$ and $N(\varepsilon^2\Delta)$.

We substitute (2.11) into the equation and boundary conditions (2.3) and act on the obtained expression with operator $\Pi_{ln}$. Owing to the indicated properties of that operator we have

$$\begin{aligned} &\Pi_{ln} L(\varepsilon^2\Delta)\mathbf{V} = L(\varepsilon^2\Delta)\mathbf{V}_{n+2} = 0 \\ &\Pi_{ln} N(\varepsilon^2\Delta)\mathbf{V}\,|_{\zeta=\pm 1} = N(\varepsilon^2\Delta)\mathbf{V}_{n+2}\,|_{\zeta=\pm 1} = \mathbf{d}^{\pm} m_n \end{aligned} \tag{2.13}$$

Since vector $\mathbf{V}_{n+2}$ is determined by (2.12), hence after its substitution into (2.13) and equating the coefficients at like $m_k(\xi)$, we obtain for the determination of $\mathbf{X}_k(\zeta)$ the recurrent system

$$\begin{aligned} &L_0\mathbf{X}_0 = 0, \quad N_0\mathbf{X}_0\,|_{\zeta=\pm 1} = 0 \\ &L_0\mathbf{X}_k - L_1\mathbf{X}_{k-1} = 0, \quad (N_0\mathbf{X}_k - N_1\mathbf{X}_{k-1})_{\zeta=\pm 1} = \delta_{2k}\mathbf{d}^{\pm} \\ &\mathbf{X}_k = (X_{k1}, X_{k2}, X_{k3}) \quad (k = 1, 2, \ldots, n+2) \end{aligned} \tag{2.14}$$

where $\delta_{2k}$ is the Kronecker delta.

The integration of system (2.14) yields

$$X_{01} = (e-\zeta)X_{02}, \quad X_{02} = D^{-1}(d_2^+ - d_2^-), \quad X_{03} = 0$$

$$X_{11} = \psi(\zeta)X_{02} - \left(X_{12}^* + \frac{e_{15}}{c_{44}} X_{13}^*\right)\zeta + X_{11}^*$$

$$X_{12} = \int_0^\zeta r(e-\zeta)\,d\zeta X_{02} + X_{12}^*, \quad X_{13} = -\int_0^\varepsilon c(e-\zeta)\,d\zeta X_{02} + X_{13}^*$$

$$X_{21} = -\int_0^\zeta \left(X_{22} + \frac{e_{15}}{c_{44}} X_{23}\right) d\zeta + \int_0^\zeta c_{44}^{-1} d\zeta \times$$

$$\int_0^\zeta (c_{11}X_{11} - c_{13}\partial X_{22} - e_{31}\partial X_{23})\,d\zeta + \int_0^\zeta c_{44}^{-1} d\zeta d_1^- + X_{21}^*$$

$$X_{22} = \int_0^\zeta F(\zeta)\,d\zeta X_{02} - X_{13}^* \int_0^\zeta \left[q\varkappa(\zeta) + r\frac{e_{15}}{c_{44}}\zeta\right] d\zeta - X_{12}^* \times$$

$$\int_0^\zeta r\zeta\,d\zeta + \int_0^\zeta r\,d\zeta X_{11}^* + d_2^- \int_0^\zeta t\,d\zeta + d_3^- \int_0^\zeta q\,d\zeta + X_{22}^*$$

$$X_{23} = \int_0^\zeta F_1(\zeta)\,d\zeta X_{02} + X_{13}^* \int_0^\zeta \left[p\varkappa(\zeta) + c\frac{e_{15}}{c_{44}}\zeta\right] d\zeta + X_{12}^* \times$$

$$\int_0^\zeta c\zeta d\zeta - X_{11}{}^* \int_0^\zeta c d\zeta + d_2{}^- \int_0^\zeta q d\zeta - d_3{}^- \int_0^\zeta p d\zeta + X_{23}{}^*$$

$$X_{11}{}^* = \frac{1}{g^{(0)}} \Big[ d_1{}^+ - d_1{}^- - \int_{-1}^{1} (c_{11}\psi(\zeta) - c_{13}F(\zeta) - e_{31}F_1(\zeta))\, d\zeta X_{02} +$$

$$\int_{-1}^{1} \left(c\varkappa(\zeta) + \frac{e_{15}}{c_{44}}\, g\zeta\right) d\zeta + X_{12}{}^* \int_{-1}^{1} g\zeta d\zeta + d_2{}^- \int_{-1}^{1} r d\zeta + d_3{}^- \int_{-1}^{1} c d\zeta \Big]$$

$$X_{13}{}^* = -\varkappa^{-1}(1)\,[d_3{}^+ - d_3{}^- - b(1) X_{02}]$$

where

$$e = \frac{g^{(1)}}{g^{(0)}}, \quad g = c_{11} - c_{13} r + e_{31} c, \quad r = \delta^{-1}(c_{13}\varepsilon_{33} + e_{31}e_{33})$$

$$c = \delta^{-1}(c_{33}e_{31} - c_{13}e_{33}), \quad \delta = \varepsilon_{33}c_{33} + e_{33}{}^2, \quad g^{(i)} = \int_{-1}^{1} g\zeta^i d\zeta$$

$$D = \int_{-1}^{1} d\zeta \int_{-1}^{\zeta} g(e-\zeta)\, d\zeta, \quad t = \delta^{-1}\varepsilon_{33}, \quad q = \delta^{-1} e_{33}, \quad p = \delta^{-1} c_{33}$$

$$\psi(\zeta) = \int_0^\zeta d\zeta \left[ \int_{-1}^{\zeta} g(e-\zeta)\, d\zeta - \int_0^\zeta r(e-\zeta)\, d\zeta + \frac{e_{15}}{c_{44}} \int_0^\zeta c(e-\zeta)\, d\zeta \right]$$

$$F(\zeta) = t \int_{-1}^{\zeta} d\zeta \int_{-1}^{\zeta} g(e-\zeta)\, d\zeta + q b(\zeta) + r\psi(\zeta)$$

$$F_1(\zeta) = q \int_{-1}^{\zeta} d\zeta \int_{-1}^{\zeta} g(e-\zeta)\, d\zeta - p b(\zeta) - c\psi(\zeta)$$

$$b(\zeta) = \int_{-1}^{\zeta} d\zeta \left[ \frac{e_{15}}{c_{44}} \int_{-1}^{\zeta} g(e-\zeta)\, d\zeta + \left(\varepsilon_{11} + \frac{e_{15}{}^2}{c_{44}}\right) \int_0^\zeta c(e-\zeta)\, d\zeta \right]$$

$$\varkappa(\zeta) = \int_{-1}^{\zeta} \left(\varepsilon_{11} + \frac{e_{15}{}^2}{c_{44}}\right) d\zeta$$

For $k > 2$ the quantities $X_{k1}, X_{k2}, X_{k3}$ are obtained from the recurrent formulas

$$X_{k1} = -\int_0^\zeta \left(X_{k2} + \frac{e_{15}}{c_{44}} X_{k3}\right) d\zeta + \int_0^\zeta c_{44}^{-1} d\zeta \times$$

$$\int_{-1}^{\zeta} (c_{11}X_{k-11} - c_{13}\partial X_{k2} - e_{31}\partial X_{k3})\, d\zeta + X_{k1}{}^*$$

$$X_{k2} = \int_0^\zeta r X_{k-11} d\zeta + \int_0^\zeta t d\zeta \int_{-1}^{\zeta} (c_{44}\partial X_{k-11} + c_{44}X_{k-12} + e_{15}X_{k-13})\, d\zeta +$$

$$\int_0^\zeta q d\zeta \int_{-1}^{\zeta} (e_{15}\partial X_{k-11} + e_{15}X_{k-12} - \varepsilon_{11}X_{k-13})\, d\zeta + X_{k2}^*$$

$$X_{k3} = -\int_0^\zeta cX_{k-11}d\zeta + \int_0^\zeta q d\zeta \int_{-1}^\zeta (c_{44}\partial X_{k-11} + c_{44}X_{k-12} + e_{15}X_{k-13})\, d\zeta - \int_0^\zeta p d\zeta \int_{-1}^\zeta (e_{15}\partial X_{k-11} + e_{15}X_{k-12} - \varepsilon_{11}X_{k-13}) d\zeta + X_{k3}^*$$

where the constants $X_{k1}^*$, $X_{k-12}^*$, $X_{k3}^*$ are determined for $k \geqslant 2$ by conditions

$$\int_{-1}^1 (c_{11}X_{k1} - c_{13}\partial X_{k+12} - e_{31}\partial X_{k+13})\, d\zeta = 0$$

$$\int_{-1}^1 (c_{44}\partial X_{k1} + c_{44}X_{k2} + e_{15}X_{k3})\, d\zeta = 0$$

$$\int_{-1}^1 (e_{15}\partial X_{k1} + e_{15}X_{k2} - \varepsilon_{11}X_{k3})\, d\zeta = 0$$

which are, however, insufficient for determining all $X_{ki}^*$ $(k = 2, 3, \ldots, n + 2)$, four of which, viz. $X_{n+21}^*$, $X_{n+22}^*$, $X_{n+23}^*$, $X_{n+12}^*$ remain arbitrary. It can be shown that solutions determined by these constants in the particular case of homogeneous biharmonic solution derived in [1].

Let us now turn to problem (2.4). We assume that

$$\tau_2^\pm = t_2^\pm m_n\ (\xi)$$

Omitting details of the derivation of solution, we present its final form

$$a_2 = \varepsilon^{-2} \sum_{s=0}^{n+1} \varepsilon^{2s} a_{2s}(\zeta)\, m_{n+1-s}\,(\xi) \tag{2.15}$$

$$a_{20} = \frac{t_2^+ - t_2^-}{c_{66}^{(0)}}, \quad a_{21} = t_2^- \int_0^\zeta c_{44}^{-1} d\zeta + a_{20} \int_0^\zeta c_{44}^{-1} d\zeta \int_{-1}^\zeta c_{66} d\zeta + a_{21}^*$$

$$a_{2s} = \int_0^\zeta c_{44}^{-1} d\zeta \int_{-1}^\zeta c_{66} a_{2s-1} d\zeta + a_{2s}^*, \quad c_{66}^{(0)} = \int_{-1}^1 c_{66} d\zeta$$

$$(s = 2, 3, \ldots, n + 1)$$

$$\left( \int_{-1}^1 c_{66} a_{2s} d\zeta = 0 \right)$$

where the constants $a_{2s}^*$ are determined by the conditions appearing there in parentheses, and the constant $a_{2n+1}^*$ remains arbitrary.

Let us indicate the singularities of formulas (2.12), (2.13), and (2.14). They are: the length of each sum is determined by the order of polyharmonic loading, each solution spontaneously expands in powers of parameter $\varepsilon$, and for small $\varepsilon$ these formulas can be simplified by rejecting small terms. The formulas for displacements, potential, voltage, and induction obviously possess these singularities, since they are obtained by the substitution of (2.12) and (2.15) into (2.1) and (1.1).

An example of formulas for displacements and the electric field potential when the plate is only subjected to a normal load, i. e.

$$q_\alpha^\pm = 0, \qquad q_3^\pm = hd_2^\pm m_n(\xi), \qquad d_3^\pm = 0$$

is provided by formulas

$$u_\alpha = \varepsilon^{-3}\left[(e-\zeta)\,\partial_\alpha w + \sum_{s=1}^{n+2} \varepsilon^{2s} X_{s1}\partial_\alpha m_{n+2-s}\right] \qquad (2.16)$$

$$u_3 = \varepsilon^{-4}\left[w + \sum_{s=1}^{n+2} \varepsilon^{2s} X_{s2} m_{n+2-s}\right]$$

$$\varphi = \varepsilon^{-2}\left[f(\zeta)\,\Delta w + \sum_{s=2}^{n+2} \varepsilon^{2s-2} X_{s3} m_{n+2-s}\right]$$

$$w = \frac{h(d_2^+ - d_2^-)}{D} m_{n+2} \qquad f(\zeta) = \int_0^\zeta c(e-\zeta)\,d\zeta - \frac{b(1)}{\varkappa(1)}$$

The first terms appearing in brackets in expressions for displacements can evidently be obtained by devising an applied theory based on the Kirchhoff-type hypotheses. Function $w$ may be considered to be the plate deflection and $\xi = e$ the neutral axis of its bending. Since solution (2.16) is exact, the remaining terms make possible the evaluation of the error of such theory, depending on various parameters of the plate. We would stress that the formulas are also valid for rapidly changing polyharmonic loads such as $x_\alpha \exp[i(mx_1 + nx_2)]$ or $\rho^k \exp(i\theta l)$, where $k$ and $l$ have the same even-values and therefore, enable us to analyze the error of the approximate theory, depending on load variation.

3. Let us now consider a composite plate consisting of alternate metal and piezoactive layers, with the metal ones forming the plate external surfaces. Each piezoactive layer may be of uneven thickness. The layers are numbered as shown in Fig. 1, with the metal layers denoted by odd subscripts $j = 1, 3, \ldots, n$ and the piezoactive ones by even subscripts $i = 2, 4, \ldots, n-1$. The thickness of each layer is denoted by $h_s$ $(s = 1, 2, \ldots)$ and the thickness of the whole plate, as previously, by $2h$. The dimensionless crosswise coordinate of the median surface of the $s$-th layer is denoted by $\zeta_s^*$. The dimensionless system of coordinates $\xi_1$, $\xi_2$, $\zeta$ is supplemented by the local system of coordinates $\xi_1$, $\xi_2$, $\zeta_s = \zeta - \zeta_s^*$ attached to each layer.

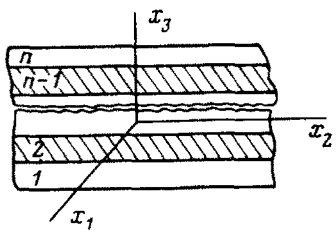

Fig. 1

Let us consider the electroelastic equilibrium of the layered element and formulate the boundary value problem. The electroelastic equilibrium of the $j$ layer is defined by Lamé equations and condition $\varphi_j = \text{const}$, while that of the $i$ layer is defined by Eqs. (1.2). The layer are assumed to be "rigidly" joined, which means that the conjugation conditions

$$u_k^{(s)}\left(\xi, \frac{\omega_s}{2}\right) = u_k^{(s+1)}\left(\xi, -\frac{\omega_{s+1}}{2}\right) \tag{3.1}$$
$$\sigma_{k3}^{(s)}\left(\xi, \frac{\omega_s}{2}\right) = \sigma_{k3}^{(s+1)}\left(\xi, -\frac{\omega_{s+1}}{2}\right) \quad (k = 1, 2, 3), \quad \omega_s = \frac{h_s}{h}$$

where the subscript $s$ corresponds to the $s$ layer, must be satisfied.

On the faces of each $i$ layer in addition to condition (3.1) the following boundary condition must be satisfied:

$$\varphi^{(i)}\left(\xi, \frac{\omega_i}{2}\right) = \varphi_{i+1}, \quad \varphi^{(i)}\left(\xi, -\frac{\omega_i}{2}\right) = \varphi_{i-1}$$

where $\varphi_{i\pm1}$ are potentials in metal layers of either the active electric field (when they are to be assumed known) or of the electric field induced in the plate by mechanical effects. In the latter case they are to be determined by the supplementary conditions

$$\iint\limits_S D_3^{(i)}\left(\xi, \pm\frac{\omega_i}{2}\right) d\xi = 0 \tag{3.2}$$

We further assume that at the faces the following stresses are applied:

$$\sigma_{k3}^{(1)}\left(\xi, -\frac{\omega_1}{2}\right) = q_k^-(\xi), \quad \sigma_{k3}^{(n)}\left(\xi, \frac{\omega_n}{2}\right) = q_k^+(\xi) \tag{3.3}$$

As previously, we assume that the right-hand sides of (3.3) satisfy the equation $\Delta^n q_k^\pm = 0$. We are not, so far, defining boundary conditions at the plate side face.

The inhomogeneous solutions of this problem can be written, after its splitting into potential and vortex parts (whose method of derivation is analgous to the expounded above), for the potential part (of the problem) as

$$u_\alpha = \varepsilon^{-3}\left[(e^* - \zeta)\,\partial_\alpha w + \sum_{k=1}^{n+2} \varepsilon^{2k} X_{k1} \partial_\alpha m_{n+2-k}\right] \quad (\alpha = 1, 2) \tag{3.4}$$

$$u_3 = \varepsilon^{-4}\left[w + \sum_{k=1}^{n+2} \varepsilon^{2k} X_{k2} m_{n+2-k}\right]$$

$$\varphi^{(i)} = \varepsilon^{-2}\left[f^{(i)}(\zeta)\,\Delta w + \sum_{k=2}^{n+2} \varepsilon^{2k-2} X_{k3}^{(i)} m_{n+2-k}\right]$$

$$\sigma_{11} = \varepsilon^{-2}\Big\{(e^* - \zeta)(c_{11}\partial_1^2 + c_{12}\partial_2^2)\,w + [cc_{1i} - \tag{3.5}$$
$$(e^* - \zeta)(c_{13}r - e_{31}c)]\,\Delta w + \sum_{k=1}^{n+2} \varepsilon^{2k}\left[X_{k1}(c_{11}\partial_1^2 + c_{12}\partial_2^2)\,m_{n+2-k} + \right.$$
$$\left.(c_{13}\partial X_{k+12} + e_{31}\partial X_{k+13})\,m_{n+1-k}\right]\Big\}$$

$$\sigma_{22} = \varepsilon^{-2}\Big\{(e^* - \zeta)(c_{12}\partial_1^2 + c_{11}\partial_2^2)\,w + [cc_{1i} -$$
$$(e^* - \zeta)(c_{13}r - e_{31}c)]\,\Delta w + \sum_{k=1}^{n+2} \varepsilon^{2k}\left[X_{k1}(c_{12}\partial_1^2 + c_{11}\partial_2^2)\,m_{n+2-k} + \right.$$
$$\left.(c_{13}\partial X_{k+12} + e_{31}\partial X_{k+13})\,m_{n+1-k}\right]\Big\}$$

$$\sigma_{33} = \int_{-1}^{\zeta} f_1(\zeta)\,d\zeta\,\Delta^2 w + \sum_{k=3}^{n+2} (c_{33}\partial X_{k2} + e_{33}\partial X_{k3} + c_{13}X_{k-11})\,m_{n+2-k}$$

$$\sigma_{12} = \varepsilon^{-2}(c_{11} - c_{12})[(e^* - \zeta)\,\partial_1\partial_2 w + \sum_{k=1}^{n+2} \varepsilon^{2k} X_{k1}\partial_1\partial_2 m_{n+2-k}]$$

$$\sigma_{\alpha 3} = \varepsilon^{-1}[-f_1(\zeta)\,\partial_\alpha \Delta w + \sum_{k=2}^{n+2} \varepsilon^{2k-2}(c_{44}X_{k2} + c_{44}\partial X_{k1} + e_{15}X_{k3})\,\partial_\alpha m_{n+2-k}]$$

$$D_\alpha = \varepsilon^{-1}[-\partial b^*(\zeta)\,\partial_\alpha \Delta w + \sum_{k=2}^{n+2} \varepsilon^{2k-2}(e_{15}X_{k2} + e_{15}\partial X_{k1} - \varepsilon_{11}X_{k3})\,\partial_\alpha m_{n+2-k}]$$

$$D_3 = \varepsilon^{-2}[-c_{1i}\Delta w + \varepsilon^2(b^*(\zeta)\,\Delta^2 w + D_{2i}m_n) + \sum_{k=3}^{n+2} \varepsilon^{2k-2}(e_{33}\partial X_{k2} - e_{31}X_{k-11} - \varepsilon_{33}\partial X_{k3})\,m_{n+2-k}]$$

where

$$w = X_{02}m_{n+2}, \quad X_{02} = \frac{h(d_2^+ - d_2^-)}{D^*}, \quad D^* = \int_{-1}^{1} f_1(\zeta)\,d\zeta$$

$$e^* = \left(g^{(1)} - \int_{-1}^{1} cs_i c_i^{(1)} d\zeta\right)\left(g^{(0)} - \int_{-1}^{1} cs_i c_i^{(0)} d\zeta\right)^{-1}, \quad c_{1i} = s_i(c_i^{(1)} - e^* c_i^{(0)})$$

$$s_i = \left(\int_{\alpha_i}^{\beta_i} p\,d\zeta\right)^{-1}, \quad c_i^{(m)} = \int_{\alpha_i}^{\beta_i} c\zeta^m d\zeta, \quad q_i^{(0)} = \int_{\alpha_i}^{\beta_i} q\,d\zeta$$

$$f_1(\zeta) = \int_{-1}^{\zeta} [cc_{1i} + g(e^* - \zeta)]\,d\zeta, \quad f^{(i)}(\zeta) = \int_{\alpha_i}^{\zeta} [pc_{1i} + c(e^* - \zeta)]\,d\zeta$$

$$D_{2i} = lX_{02} + s_i c_i^{(1)} X_{12}^* - s_i c_i^{(0)} X_{11}^* + s_i q_i^{(0)} d_2^- - (\varphi_{i+1} - \varphi_{i-1})\,s_i$$

$$l = s_i \int_{\alpha_i}^{\beta_i} \left[q \int_{-1}^{\zeta} f_1(\zeta)\,d\zeta - pb^*(\zeta) - c\psi^*(\zeta)\right] d\zeta$$

$$\psi^*(\zeta) = \int_0^{\zeta} \left[f_1(\zeta) + \frac{e_{15}}{c_{44}} f^{(i)}(\zeta) - \int_0^{\zeta} (r(e^* - \zeta) + qc_{1i})\,d\zeta\right] d\zeta$$

$$b^*(\zeta) = \int_{-1}^{\zeta} \left[\frac{e_{15}}{c_{44}} f_1(\zeta) + \left(\varepsilon_{11} + \frac{e_{15}^2}{c_{44}}\right) f^{(i)}(\zeta)\right] d\zeta,$$

$$X_{11} = \psi^*(\zeta)\,X_{02} - X_{12}^*\zeta + X_{11}^*$$

$$X_{12} = \int_0^{\zeta} [r(e^* - \zeta) + qc_{1i}]\,d\zeta X_{02} + X_{12}^*, \quad X_{13}^{(i)} = -f^{(i)}(\zeta)\,X_{02}$$

$$\alpha_i = \zeta^*_i - \omega_i/2, \quad \beta_i = \zeta_i + \omega_i/2$$

The remaining $X_{k1}$, $X_{k2}$, $X_{k3}$ are determined by the recurrent formulas similar to those in the preceding problem.

The inhomogeneous solutions of the vortex problem are the same as the corresponding solutions of the preceding problem.

Let us illustrate the application of the derived formulas on two simple examples.

E x a m p l e 1. Let us consider the electroelastic equilibrium of a plane three-layer element whose external metal layers are of thickness $h_1$, and the two inner piezoceramic layers are of the same thickness $h_2 / 2$ each and differ by the directions of their initial polarization vectors. We assume the element to be deformed by the applied potential difference $\Delta\varphi = \varphi_3 - \varphi_1$.

In this case it is necessary to set in formulas (3.4) and (3.5) $n = 0,\ q_k^{\pm} = 0$ which yields (for $w = X_{02} = 0$), the equalities

$$m_1 = A_3\xi_1^3 + A_2\xi_1^2 + A_1\xi_1 + A_0,\ m_0 = -6A_3\xi_1 - 2A_2$$

where $A_k$ are arbitrary constants determined by the boundary conditions when $\xi_1 = \pm 1$. We assume, for definiteness, that at the element ends the following conditions are specified:

$$\int_{-1}^{1} \sigma_{11}\zeta\, d\zeta = \int_{-1}^{1} D_1\, d\zeta = \int_{-1}^{1} u_3\, d\zeta = 0$$

Then

$$u_1 = \varepsilon^{-1}hB\Delta\varphi\zeta\xi_1,\ u_3 = {}^1/_2\varepsilon^{-2}hB\Delta\varphi\,(1 - \xi_1^2)$$

$$\varphi = \left[B\int_{-1+\omega_1}^{\zeta} c\zeta\, d\zeta - pK\left(\zeta + \frac{\omega_2}{2}\right)\right]\Delta\varphi + \varphi_1$$

$$\sigma_{11} = (Bg\zeta - Kc)\,\Delta\varphi,\quad D_3 = K\Delta\varphi,\quad \sigma_{13} = D_1 = 0$$

$$B = \frac{12cp^{-1}\omega_2}{(3c^2p^{-1} - 4g + 4\beta)\,\omega_2^3 - 32\beta}$$

$$K = \frac{1}{p\omega_2}\left(\frac{1}{4}cBc\omega_2^2 - 1\right),\ \beta = \frac{4\mu\,(\lambda + \mu)}{\lambda + 2\mu}$$

where $\lambda$ and $\mu$ are Lamé coefficients for the material of electrodes.

E x a m p l e 2. We assume the described element to be deformed by bending moments $M$ applied to its ends. For the induced potential difference using formulas (3.4) and (3.5) with condition (3.2) we, then, have

$$\Delta\varphi = \frac{1}{h^2}\frac{3c\omega_2^2M}{8\beta + (g - \beta)\,\omega_2^3}$$

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